Weak convergence of Euler-Maruyama's approximation for SDEs under integrability condition

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Consider

$$\mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t),$$

where $b \in \mathbb{L}_p^q := L^q(\mathbb{R}_+; L^p(\mathbb{R}^d))$ with $p,\ q > 1,\ \frac{d}{p} + \frac{2}{q} < 1$.

- Krylov-Röckner (2005), existence and uniqueness of strong solution when $\sigma(t, x) = Id$.
- Zhang, X.C. (2005, 2011), considered the case of multiplicative noise.
- Basic methods: Krylov's estimate and Zvonkin's tranform

- Wang, F.Y. (2017) Ann. Probab. used Harnack inequality to discuss the *nonexplosion* of SDEs with singular drifts and its existence, uniqueness and regularity of invariant probability measures.
- Wang, F.Y. (2018) PTRF, extended this method to deal with degenerate SDEs and path-dependent SDEs.
- The drift $b(t,x)$ may not belong to $\mathbb{L}_p^q$ or satisfy any Lyapunov type condition.

Example

$$b(t,x) = \left\{ \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{|x-n|^2}\right) \right\}^{\frac{1}{2}} - x, \quad x \in \mathbb{R}. \qquad \text{(E1)}$$

Euler-Maruyama's approximation

Numerical approximation plays important role in the application of SDEs.

- Under global Lipschitz condition, Strong convergence of Euler-Maruyama's (EM's) approximation, cf. Kloeden and Platen (1992).
- Under one-sided Lipschitz condition, Strong convergence of EM's approximation, cf. Higham, Mao, Stuart (2002).

Generally, the linear growth condition plays a crucial role, because

♠ Hutzenthaler, Jentzen, Kloeden (2011, 2013) showed that EM's approximation may diverge to ∞ both in strong and weak sense if the coefficients grow superlinearly.

Convergence of EM's approximation

The convergence of EM's approximation has been investigated for various criteria:

- Convergence of expectation of functionals of solutions of SDEs, cf. Talay and Tubaro (1990)
- Convergence of distribution function, cf. Bally and Talay (1996)
- Convergence in Wasserstein distance, cf. e.g. Alfonsi et al. (2014)

- Yan (2002) Ann. Probab. possibly discontinuous coefficients but satisfying linear growth condition;
- Kohatsu-Higa et al. (2012), bounded Hölder continuous drifts;
- Ngo and Taguchi (2018), Hölder continuous drifts and satisfying the sub-linear growth condition.

In this talk, we shall study the weak convergence in the form

$$\mathbb{E}[f(X_\delta(t))] \to \mathbb{E}[f(X(t))], \quad \text{as } \delta \to 0, \quad \forall \text{ bounded } f,$$

and in the **W**asserstein distance.

Two main issues

First, since singular $b(t,x)$ may not be well-defined for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, in order to define the EM's approximation for every initial value, certain regularization is needed. For example, consider the drift $b(t,x)$ defined previous in (E1), we define

$$Z(t,x) = \Big\{ \sum_{n=1}^{d} \log\Big(1 + \frac{1}{|x-n|^2}\Big)\Big\}^{\frac{1}{2}}, \quad \psi_\varepsilon = \varepsilon^{-d}\big(\psi(x/\varepsilon)\big),$$

where $\psi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}_+)$ with $\int \psi(y)\mathrm{d}y = 1$.

$$b_\varepsilon(t,x) = \big(Z(t,\cdot) * \psi_\varepsilon\big)(x) - x.$$

Correspondingly, *define* $\mathrm{d}X_\varepsilon(t) = b_\varepsilon(t, X_\varepsilon(t))\mathrm{d}t + \mathrm{d}W(t).$

- ♠ Estimate the difference between $X(t)$ and $X_\varepsilon(t)$.

Two main issues

For the first problem, when b and $\tilde{b}$ are in $\mathbb{L}_p^q$,
Zhang, X.C. (2016) Ann. Appl. Probab. has proved

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|X^b(t)-X^{\tilde{b}}(t)\big|^2\Big]\le C\|b-\tilde{b}\|^2_{\mathbb{L}_p^q}.$$

- This result is based on Zvonkin's transform, which cannot deal with b given by (E1).
- This result cannot deal with degenerate SDEs.

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- This result is based on Zvonkin's transform, which cannot deal with b given by (E1).
- This result cannot deal with degenerate SDEs.

Second, define EM's approximation

$$\mathrm{d}X_\varepsilon^\delta(t) = b_\varepsilon([t/\delta]\delta, X_\varepsilon^\delta([t/\delta]\delta))\mathrm{d}t + \mathrm{d}W(t).$$

♠♠ Estimate the difference between $X_\varepsilon(t)$ and $X_\varepsilon^\delta(t)$.

Contents

Consider the SDE

$$\mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sigma \mathrm{d}W(t), \quad X(0) = x_0 \in \mathbb{R}^d, \tag{E2}$$

where $(W(t))$ d-dim B.M., $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times d}$.

(H_σ) $\exists\, \lambda > 0$ s.t. $\lambda^{-1}|x|^2 \le |\sigma x|^2 \le \lambda |x|^2$, $\forall\, x \in \mathbb{R}^d$.

For $V \in C^2(\mathbb{R}^d)$, define

$$\mu_0(\mathrm{d}x) = \mathrm{e}^{-V(x)}\mathrm{d}x, \quad Z_0 = -\sum_{i,j=1}^{d} (a_{ij}\partial_j V)e_i,$$

where $\{e_i\}_{i=1}^d$ canonical orthonormal basis of $\mathbb{R}^d$, $(a_{ij}) = \sigma\sigma^*$.

Define a class of functions:

$$\mathscr{V} = \left\{ V \in C^2(\mathbb{R}^d) \middle| \begin{array}{l} \mu_0(\mathbb{R}^d) = 1, \ \exists\, K_0 > 0, \\ |Z_0(x) - Z_0(y)| \leq K_0 |x-y| \ \ \forall\, x, y \in \mathbb{R}^d \end{array} \right\}.$$

The Wasserstein distance between two probability measures μ and ν on $\mathbb{R}^d$:

$$W_1(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \, \pi(\mathrm{d}x, \mathrm{d}y); \ \pi \in \mathscr{C}(\mu, \nu) \right\},$$

where $\mathscr{C}(\mu, \nu)$ the collection of all couplings of μ and ν.

Theorem 1. Let $(\widetilde{X}(t))$ be the solution to the following SDE:

$$\mathrm{d}\widetilde{X}(t) = \tilde{b}(t, \widetilde{X}(t))\mathrm{d}t + \sigma \mathrm{d}W(t), \quad \widetilde{X}(0) = x_0.$$

Assume (H_σ) holds. Let $T > 0$ be given. Assume $\exists$ $V \in \mathscr{V}$ such that Z_0, $Z(t,x) := b(t,x) - Z_0(x)$, $\widetilde{Z}(t,x) := \tilde{b}(t,x) - Z_0(x)$ satisfy:

(H1) $\exists$ a constant $\eta > 2\lambda T d$ such that

$$\sup_{t\in[0,T]} \mu_0\Big(\mathrm{e}^{\eta|Z(t,\cdot)|^2}\Big) < \infty, \quad \sup_{t\in[0,T]} \mu_0\Big(\mathrm{e}^{\eta|\widetilde{Z}(t,\cdot)|^2}\Big) < \infty.$$

Then, $\forall$ $\xi > d$, $\exists$ a constant $C = C(K_0, T, \lambda, \xi, \eta)$ such that

$$\sup_{t\in[0,T]} W_1(\mathscr{L}(X(t)), \mathscr{L}(\widetilde{X}(t))) \le C\Big\{\int_0^T \frac{\mu_0(|Z-\widetilde{Z}|^{q_0\xi}(s,\cdot))^{\frac{1}{\xi}}}{(1-\mathrm{e}^{-K_0 s})^{\frac{d}{\xi}}}\mathrm{d}s\Big\}^{\frac{1}{q_0}},$$

where $q_0 = p_0/(p_0-1)$, $p_0 = \sqrt{\frac{\eta}{2\lambda T d}} \wedge 2$.

Assume the drift b in (E2) is well-defined for every (t, x), then the EM's approximation of (E2) is defined by

$$\mathrm{d}X_\delta(t) = b(t_\delta, X_\delta(t_\delta))\mathrm{d}t + \sigma \mathrm{d}W(t), \quad X_\delta(0) = x_0,$$

where $t_\delta = [t/\delta]\delta$ for $\delta > 0$.

Theorem 2. Assume (H_σ) holds. $T > 0$ be given. Assume $\exists$ $V \in \mathscr{V}$ such that $(t,x) \mapsto Z(t,x) := b(t,x) - Z_0(x)$ is continuous. Suppose

(H2) $\exists$ $\eta_0 > 0$ and $\eta > 4\lambda T d$ such that

$$\mu_0\left(\mathrm{e}^{\eta_0|Z_0|^2}\right) < \infty, \quad \sup_{t\in[0,T]} \mu_0\left(\mathrm{e}^{\eta|Z(t,\cdot)|^2}\right) < \infty.$$

Then,

$$\lim_{\delta\to 0} \mathbb{E}[f(X_\delta(t))] = \mathbb{E}[f(X(t))], \quad t \in [0,T], \forall\, f \in \mathscr{B}_b(\mathbb{R}^d),$$

and

$$\lim_{\delta\to 0} W_1(\mathscr{L}(X_\delta(t)), \mathscr{L}(X(t))) = 0, \quad t \in [0,T].$$

When $b(t,x) = b(x)$ is time homogeneous, conditions (H1) and (H2) can be replaced respectively by (H1') and (H2') below, and the corresponding results are still valid.

$(\mathrm{H1}')$ $\exists\, \eta > 0$ such that

$$\mu_0\big(\mathrm{e}^{\eta|Z|^2}\big) < \infty \quad \text{and} \quad \mu_0\big(\mathrm{e}^{\eta|\widetilde{Z}|^2}\big) < \infty.$$

$(\mathrm{H2}')$ $\exists\, \eta > 0$ such that

$$\mu_0\big(\mathrm{e}^{\eta|Z_0|^2}\big) < \infty \quad \text{and} \quad \mu_0\big(\mathrm{e}^{\eta|Z|^2}\big) < \infty.$$

In Theorem 1, now the estimate is:

$$\sup_{t\in[0,T]} W_1(\mathscr{L}(X(t)), \mathscr{L}(\widetilde{X}(t))) \leq C\mu_0\big(|Z-\widetilde{Z}|^{2\xi}\big)^{\frac{1}{2\xi}}.$$

Theorem 3. Suppose the conditions of Theorem 2 hold. In addition, assume that $\exists\ K_1,\ m_1 > 0,\ \alpha \in (0,1]$ and a polynomially bounded function $h : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$|Z(t,x) - Z(t,y)| \le K_1(1 + |x|^{m_1} + |y|^{m_1})|x-y|, t \in [0,T],\ x,y \in \mathbb{R}^d,$$
$$|Z(t,x) - Z(s,x)| \le h(x)|t-s|^{\alpha}, \qquad t,s \in [0,T],\ x \in \mathbb{R}^d.$$

Then

$$W_1(\mathscr{L}(X_\delta(t)), \mathscr{L}(X(t))) \le C\delta^{\frac{1}{2}\wedge\alpha}, \quad t \in [0,T].$$

Example

$$b(t,x)=\left\{\sum_{n=1}^{\infty}\log\left(1+\frac{1}{|x-n|^2}\right)\right\}^{\frac{1}{2}}-x,\quad x\in\mathbb{R}. \tag{E1}$$

Take $V(x)=x^2/2+\log(\sqrt{2\pi})$, then $\mu_0(\mathrm{d}x)=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x$, $Z_0(x)=-x$, and

$$Z(x)=\left\{\sum_{n=1}^{\infty}\log\left(1+\frac{1}{|x-n|^2}\right)\right\}^{\frac{1}{2}}.$$

Then for any $\eta>0$,

$$\mu_0\left(\mathrm{e}^{\eta|Z|^2}\right)<\infty.$$

Let $X_\varepsilon(t)$ and $X_\varepsilon^\delta(t)$ be determined previously. Then, for every $T>0$, $\xi>1$,

$$W_1(\mathscr{L}(X_\varepsilon(t)),\mathscr{L}(X(t)))\le C\left(\mu_0(|Z-Z_\varepsilon|^{2\xi})\right)^{\frac{1}{2\xi}},$$

$$W_1(\mathscr{L}(X_\varepsilon^\delta(t)),\mathscr{L}(X_\varepsilon(t)))\le C\delta^{\frac{1}{2}},\qquad t\in[0,T].$$

Contents

Consider

$$\begin{aligned} \mathrm{d}X^{(1)}(t) &= X^{(2)}(t)\mathrm{d}t, \\ \mathrm{d}X^{(2)}(t) &= b(t, X^{(1)}(t), X^{(2)}(t))\mathrm{d}t + \sigma \mathrm{d}W(t). \end{aligned}$$

$b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Assume b is well-defined everywhere. Consider its EM's approximation:

$$\begin{aligned} \mathrm{d}X_\delta^{(1)}(t) &= X_\delta^{(2)}(t)\mathrm{d}t, \\ \mathrm{d}X_\delta^{(2)}(t) &= b(t_\delta, X_\delta^{(1)}(t_\delta), X_\delta^{(2)}(t_\delta))\mathrm{d}t + \sigma \mathrm{d}W(t), \end{aligned}$$

Theorem

Let $T > 0$ be given. Assume that (H_σ) *holds and there exists* $V \in \mathscr{V}$ *such that* $(t, x_1, x_2) \mapsto Z(t, x_1, x_2) := b(t, x_1, x_2) - Z_0(x_2)$ *is continuous. In addition, assume that*

(A2) *there exist constants* $\eta_0 > 0$, $\eta > 4\lambda T d$ *such that*

$$\mu_0\big(\mathrm{e}^{\eta_0 |Z_0|^2}\big) < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \mu_0\Big(\sup_{x_1 \in \mathbb{R}^d} \mathrm{e}^{\eta |Z(t,x_1,\cdot)|^2}\Big) < \infty.$$

Then,

$$\lim_{\delta\to 0} W_1\big(\mathscr{L}(X_\delta^{(1)}(t), X_\delta^{(2)}(t)), \mathscr{L}(X^{(1)}(t), X^{(2)}(t))\big) = 0.$$

Introduce a reference process

$$\mathrm{d}Y(t) = Z_0(Y(t))\mathrm{d}t + \sigma \mathrm{d}W(t).$$

Then, use $(Y(t))$ to present a new representation of the studied SDE and its EM's approximation:

$$\begin{aligned}\mathrm{d}Y(t) &= b(t, Y(t))\mathrm{d}t + \sigma\big(\mathrm{d}W(t) - \sigma^{-1}b(t, Y(t))\mathrm{d}t + \sigma^{-1}Z_0(Y(t))\mathrm{d}t\big)\\ &= b(t, Y(t))\mathrm{d}t + \sigma \mathrm{d}\widehat{W}_1(t),\end{aligned} \tag{E3}$$

Define a new probability measure

$$Q_1 := \exp\left[\int_0^T \langle \sigma^{-1}(Z(s, Y(s))), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}(Z(s, Y(s)))|^2 \mathrm{d}s\right]$$

if

$$\mathbb{E}\exp\left[\frac{1}{2}\int_0^T |\sigma^{-1}(Z(s, Y(s)))|^2 \mathrm{d}s\right] < \infty.$$

Similarly, rewrite

$$\mathrm{d}Y(t) = b(t_\delta, Y(t_\delta))\mathrm{d}t + \sigma \mathrm{d}\widehat{W}_2(t), \tag{E4}$$

where

$$\widehat{W}_2(t) = W(t) + \int_0^t \sigma^{-1}(Z_0(Y(s)) - b(s_\delta, Y(s_\delta)))\mathrm{d}s, \ \ t \in [0, T].$$

Define a new probability measure

$$Q_2 = \exp\Big[- \int_0^T \langle \sigma^{-1}(Z_0(Y(s)) - b(s_\delta, Y(s_\delta))), \mathrm{d}W(s)\rangle \\ - \frac{1}{2}\int_0^T |\sigma^{-1}(Z_0(Y(s)) - b(s_\delta, Y(s_\delta)))|^2 \mathrm{d}s\Big]\mathbb{P},$$

if

$$\mathbb{E}\exp\Big[\frac{1}{2}\int_0^T |\sigma^{-1}(Z_0(Y(s)) - b(s_\delta, Y(s_\delta)))|^2 \mathrm{d}s\Big] < \infty.$$

Key point of the proof

Lemma

Let $G : \mathbb{R}^d \to \mathbb{R}_+$ *be a measurable function. If there exists a constant* $\eta > 0$ *such that* $\mu_0(\mathrm{e}^{\eta G}) < \infty$*, then, for any* $\beta, T > 0$*,*

$$\mathbb{E}\left[\mathrm{e}^{\beta \int_0^T G(Y(s))\mathrm{d}s}\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[\mathrm{e}^{\beta \int_0^T G(Y(s_\delta))\mathrm{d}s}\right] < \infty.$$

Key point of the proof

$$
\begin{aligned}
|\mathbb{E}_{\mathbb{P}} f(X(t)) - \mathbb{E}_{\mathbb{P}} f(X_\delta(t))| &= |\mathbb{E}_{Q_1} f(Y(t)) - \mathbb{E}_{Q_2} f(Y(t))| \\
&= \left| \mathbb{E}_{\mathbb{P}} \left\{ f(Y(t)) \left(\frac{\mathrm{d}Q_1}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}Q_2}{\mathrm{d}\mathbb{P}} \right) \right\} \right| \\
&\leq \|f\|_\infty \mathbb{E}_{\mathbb{P}} \left| \frac{\mathrm{d}Q_1}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}Q_2}{\mathrm{d}\mathbb{P}} \right|.
\end{aligned}
$$

Thank You !

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